

GORENSTEIN ALGEBRAS PRESENTED BY QUADRICS

JUAN MIGLIORE* AND UWE NAGEL⁺

ABSTRACT. We establish restrictions on the Hilbert function of standard graded Gorenstein algebras with only quadratic relations. Furthermore, we pose some intriguing conjectures and provide evidence for them by proving them in some cases using a number of different techniques, including liaison theory and generic initial ideals.

1. INTRODUCTION

An important aspect of the Eisenbud-Green-Harris conjectures related to the Cayley-Bacharach theorem [7] is the notion of artinian Gorenstein algebras that are quotients of complete intersections of quadrics. An important class of such algebras is one where the defining ideal is actually generated by quadrics. We will say that such an algebra is *presented by quadrics*. In this paper we study such algebras. Ideals generated by quadrics have been of interest over the years in many settings, for example, as homogeneous ideals of sufficiently positive embeddings of smooth projective varieties ([8]), as Stanley-Reisner ideals of simplicial flag complexes ([23]), or in studies of projective dimensions ([3]).

Let $R = k[x_1, \dots, x_r]$, where k is a field. Our goal in this paper is twofold. First and foremost, we are interested in analyzing the consequences on the h -vector (i.e. Hilbert function) $\{h_i\}$ when the Gorenstein ideal is generated by quadrics. Specifically, what are the connections between the embedding dimension (= codimension in the artinian case) $r = h_1$, the socle degree e (i.e. the last degree in which $h_i \neq 0$), and the value h_2 of the h -vector in degree 2 (equivalently, the number of quadric generators)? Our second goal is to give constructions of such algebras, in a first attempt to determine to what extent the limitations given on h_1, h_2 and e give a complete picture. Some of our results are actually given for algebras where the defining ideal merely contains a complete intersection of quadrics. However, our main focus is where all the minimal generators are quadrics.

The condition that an artinian Gorenstein algebra be presented by quadrics is obviously very restrictive, and one naturally would like to know how restrictive it is, for instance on the Hilbert function. Even under the weaker assumption that I contains a regular sequence of quadrics, it is not hard to see that $e \leq r$, with equality if and only if R/I is a complete intersection of quadrics (Proposition 2.3). In this case, $h_i = \binom{r}{i}$, so in particular $h_2 = \binom{r}{2}$. Is it true that for other values of e and r there is only one possible Hilbert function? Note first that the artinian condition forces $h_2 \leq \binom{r}{2}$, since $\binom{r+1}{2} - r = \binom{r}{2}$.

Now assume that R/I is presented by quadrics. We show that if $4 \leq e \leq r - 2$, there are always at least two possible Hilbert functions (Proposition 2.13), by focusing on the possibilities for h_2 . A theorem of Kunz gives that $h_2 = \binom{r}{2} - 1$ is impossible for artinian Gorenstein algebras presented by quadrics.

* Part of the work for this paper was done while the first author was sponsored by the National Security Agency under Grant Number H98230-07-1-0036 and H98230-09-1-0031.

⁺ Part of the work for this paper was done while the second author was sponsored by the National Security Agency under Grant Number H98230-07-1-0065 and H98230-09-1-0032.

On the other hand, when $e = r - 1$ and R/I is presented by quadrics, we show that it is still true that the Hilbert function is uniquely determined (Theorem 3.1). In particular, we must have $h_2 = \binom{r}{2} - 2$. (We believe that the converse holds as well, but have not found a proof.) In Theorem 3.1 we also show that if $e = r - 1$ and the ideal is only assumed to contain a complete intersection of r quadrics, then there are exactly three possible values of h_2 : $\binom{r}{2}$, $\binom{r}{2} - 1$ and $\binom{r}{2} - 2$. The latter is the *only* possibility if R/I is presented by quadrics, but the former two show that the Hilbert function is not uniquely determined in the more general setting, when $e = r - 1$. This result is enough to give a complete classification of the possible h -vectors for artinian Gorenstein algebras presented by quadrics when $r \leq 5$ (Proposition 3.3).

A great deal of interest has been shown recently in the Weak Lefschetz Property. We conjecture that in characteristic zero, an artinian Gorenstein algebra presented by quadrics has this property (Conjecture 4.5). A much weaker condition is that the homomorphism induced multiplication by a general linear form from degree 1 to degree 2 be injective (when the socle degree is ≥ 3). We begin section 4 by posing our Injectivity Conjecture (Conjecture 4.1), which says that this property should always hold for an artinian ideal presented by quadrics, of socle degree ≥ 3 . If this conjecture is true, it has many consequences given in this section and the next. We prove that the Injectivity Conjecture holds for a complete intersection of quadrics in characteristic zero (Proposition 4.3), using the Socle Lemma [12]. In the case of socle degree 4, we also give an upper bound on h_2 for artinian Gorenstein algebras presented by quadrics for which the Injectivity Conjecture holds (Corollary 4.9).

When the socle degree is $e = 3$, clearly for each r the only possible Hilbert function for an artinian Gorenstein algebra is $(1, r, 1)$. We show in Example 2.9 that for each $r \geq 3$ such a Hilbert function does occur for an artinian Gorenstein algebra presented by quadrics. Conversely, we pose the “ $h_2 = r$ Conjecture” (Conjecture 5.1), which implies that if $h_2 = r$ and R/I is presented by quadrics, then $e = 3$. We prove most of this conjecture (Proposition 5.2) for algebras satisfying the Injectivity Conjecture, and in particular we obtain the implication just mentioned. For the remainder of section 5, using the theory of generic initial ideals, we show that for Gorenstein algebras presented by quadrics, with $r < 4e - 6$, if $h_2 = r$ then the Injectivity Conjecture holds (Theorem 5.13). As a corollary, we obtain the result that $h_2 = r$ is equivalent to $e = 3$ for $3 \leq r \leq 9$.

In the course of proving these results we give some constructions of artinian Gorenstein algebras presented by quadrics, using tensor products of algebras and using inverse systems. The main tool in this paper, however, is liaison theory.

2. FIRST CONSTRUCTIONS AND HILBERT FUNCTIONS

Let $R = k[x_1, \dots, x_r]$, where k is a field (where necessary we will add assumptions about k). Let $A = R/I$ be a standard graded artinian Gorenstein algebra of socle degree e , not necessarily a complete intersection. Assume that I does not contain any elements of degree 1. Consider the conditions

- (1) the minimal generators of I all have degree 2;
- (2) I contains a regular sequence of r quadrics, but also possibly minimal generators of degree ≥ 3 .

In this paper we consider the following questions. Under each of these conditions (separately), what are the possible Hilbert functions for R/I ? What are the possible minimal free resolutions for R/I ? When does R/I necessarily have the Weak Lefschetz Property?

Recall that a graded algebra A has the *Weak Lefschetz Property* if the multiplication $\times L : [A]_{i-1} \rightarrow [A]_i$ by a general form L has maximal rank for each i . The algebra A is said to have the Strong Lefschetz property if, for each $d \geq 1$ and each i , the multiplication $\times L^d : [A]_{i-d} \rightarrow [A]_i$ has maximal rank.

Definition 2.1. An algebra of type (1) above will be referred to as a (*standard graded artinian*) *Gorenstein algebra presented by quadrics*. An algebra of type (2) will be referred to as a (*standard graded artinian*) *Gorenstein algebra containing a regular sequence of r quadrics*. We stress that it is assumed that I contains no linear form. Let $\underline{h} = (1, r, h_2, \dots, h_{e-1} = r, h_e = 1)$ be the h -vector of A . Since A is artinian, this coincides with the Hilbert function of A , and we generally use the latter terminology (except where we consider non-artinian Gorenstein algebras). We call e the *socle degree* of A .

Remark 2.2. The possible Hilbert functions of Gorenstein algebras presented by quadrics do not coincide with those of Gorenstein algebras containing a regular sequence of r quadrics. For example, consider the case $n = 6$ and the complete intersection $R/I = R/(x_1^2, \dots, x_6^2)$. This algebra has Hilbert function $(1, 6, 15, 20, 15, 6, 1)$. It is also known that this algebra has the WLP, and even the SLP [22], [24], [21]. Hence if L is a general linear form, the Hilbert function of the Gorenstein algebra $R/(I : L)$ is $(1, 6, 15, 15, 6, 1)$. This Hilbert function implies that $I : L$ contains no linear forms and precisely six independent quadrics, namely the minimal generators x_1^2, \dots, x_6^2 of I . Since $I : L \neq I$, it must have minimal generators in higher degree. In fact it must have $5 = 20 - 15$ cubic generators, and a priori possibly generators of higher degree. However, since the six quadrics form a complete intersection, they have no linear syzygy. This observation, even together with duality, is not quite enough to deduce the minimal free resolution. However, it is a simple matter to compute it on a computer algebra program, and indeed we first produced it using CoCoA [5]:

$$\begin{aligned} 0 \rightarrow R(-11) \rightarrow R^5(-8) \oplus R^6(-9) \rightarrow R^5(-6) \oplus R^{36}(-7) \rightarrow \\ R^{31}(-5) \oplus R^{31}(-6) \rightarrow R^{36}(-4) \oplus R^5(-5) \rightarrow \\ R^6(-2) \oplus R^5(-3) \rightarrow R \rightarrow R/J \rightarrow 0 \end{aligned}$$

It is clear that no Gorenstein algebra presented by quadrics can have this Hilbert function or (consequently) this resolution.

The general problem addressed in this paper is to make a first study of the possible Hilbert functions of such algebras. Of course the socle degree is an important part of this question, and since the ideal is generated by quadrics, and the number of minimal generators can be read from h_2 , we are interested in the value h_2 as well. Our goal is thus to say as much as possible about the interconnections between h_2 , r and e . We first make the following observation.

Proposition 2.3. *Let $A = R/I$ be an artinian Gorenstein algebra containing a regular sequence of r quadrics, with socle degree e and h -vector $(1, r, h_2, \dots, h_{e-1} = r, h_e = 1)$. Let ν be the number of quadratic minimal generators of I . Then*

- (1) $h_2 = \binom{r+1}{2} - \nu$.
- (2) $\nu \geq r$ since A is artinian.
- (3) $e = r$ if and only if A is a complete intersection of quadrics (i.e. $\nu = r$ and I has no generators of higher degree). In this case

$$h = \left(1, \quad r, \quad \binom{r}{2}, \quad \binom{r}{3}, \quad \dots, \quad \binom{r}{r-3}, \quad \binom{r}{r-2}, \quad r, \quad 1 \right)$$

- (4) If A is not a complete intersection then $e < r$.
- (5) If we further assume that all minimal generators are quadrics, then no such algebra exists for $\nu = r + 1$ (i.e. $h_2 = \binom{r}{2} - 1$).

Proof. Parts (1) through (4) are the result of a simple computation, plus the fact that if I is generated by quadrics then it contains a complete intersection of r quadrics. Part (5) follows from Kunz's theorem [14], since if $\nu = r + 1$ then I is an almost complete intersection, which is never Gorenstein. \square

In order to conveniently state the next result, we will use h -polynomials. If A is an artinian algebra, then its Hilbert series is a polynomial $h(A)$ that is called the h -polynomial of A . Its coefficients form the h -vector of A .

Proposition 2.4. *If A and B are artinian Gorenstein algebras presented by quadrics, then so is $A \otimes_k B$, and its h -polynomial is*

$$h(A \otimes_k B) = h(A) \cdot h(B).$$

This is an immediate consequence of the following more general observation.

Lemma 2.5. *Let A and B be standard graded algebras. Then:*

- (a) *The h -polynomial of $A \otimes_k B$ is $h(A) \cdot h(B)$.*
- (b) *The Cohen-Macaulay type of $A \otimes_k B$ is the product of the Cohen-Macaulay types of A and B .*
- (c) *If A and B are presented by quadrics, then so is $A \otimes_k B$.*

Proof. (a) follows because the Hilbert function of $A \otimes_k B$ is

$$h_{A \otimes_k B}(m) = \sum_{i+j=m} h_A(i) \cdot h_B(j).$$

(b) and (c) are true because the minimal free resolution of $A \otimes_k B$ is the tensor product of the minimal free resolutions of A and B . \square

Remark 2.6. The part about the Gorensteinness of $A \otimes_k B$ in Proposition 2.4 can also be seen using inverse systems. In fact, if A and B are the annihilators of $f \in R$ and $g \in R$, respectively, then $A \otimes_k B$ is the annihilator of $f \otimes_k g \in R \otimes_k R$.

As a first consequence we note:

Corollary 2.7. *If (h_0, \dots, h_e) is the h -vector of a Gorenstein algebra presented by quadrics, then so is the h -vector $(h_0, h_0 + h_1, h_1 + h_2, \dots, h_{e-1} + h_e, h_e)$.*

Proof. Let $A = R/I$ be a standard graded Gorenstein algebra with the given h -vector (h_0, \dots, h_e) , where the ideal I is generated by quadrics. Put $B = k[x]/x^2$. Then $A \otimes_k B$ is Gorenstein with the desired properties. \square

We saw in Proposition 2.3 that for an artinian Gorenstein algebra presented by quadrics, the socle degree e is at most equal to r . The case $e = 1$ is trivial (it is a quotient of $k[x]$). Now we show that all values in between do occur.

Corollary 2.8. *Fix $r \geq 2$. An artinian Gorenstein algebra presented by quadrics and having codimension r exists with any socle degree e satisfying $2 \leq e \leq r$.*

Proof. Note that if $e = 1$ we (trivially) have $r = 1$. One has only to begin with artinian Gorenstein algebras with h -vector $(1, s, 1)$, with $2 \leq s \leq r - 1$, and apply the above construction. It is easily shown (and well-known) that such algebras are generated by quadrics, using the basic properties of the shifts in the minimal free resolutions of artinian Gorenstein algebras. See also Example 2.9. \square

Example 2.9. Let $r \leq 8$. We immediately obtain the following h -vectors corresponding to Gorenstein algebras presented by quadrics.

Group 0	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
1 2 1	1 3 1	1 4 1	1 5 1	1 6 1	1 7 1	1 8 1
1 3 3 1	1 4 4 1	1 5 5 1	1 6 6 1	1 7 7 1	1 8 8 1	
1 4 6 4 1	1 5 8 5 1	1 6 10 6 1	1 7 12 7 1	1 8 14 8 1		
1 5 10 10 5 1	1 6 13 13 6 1	1 7 16 16 7 1				
1 6 15 20 15 6 1	1 7 19 26 19 7 1	1 8 23 32 23 8 1				
1 7 21 35 35 21 7 1	1 8 26 45 45 26 8 1					
1 8 28 56 70 56 28 8 1						

Notice that in Group 0 we have precisely the Hilbert functions of complete intersections of quadrics, and in fact by Proposition 2.3 any Gorenstein algebra presented by quadrics with such Hilbert function must be a complete intersection. More generally, for $i \geq 0$, in Group i we have socle degree $e = r - i$ and $h_2 = \binom{r}{2} - \binom{i+2}{2} + 1$. We will see in the next section that Group 1 is also special: for artinian Gorenstein algebras presented by quadrics, the condition $e = r - 1$ uniquely determines the Hilbert function. (Proposition 3.1 gives this and more.)

The question naturally arises at this point whether the lists given in Example 2.9 contain all possible Hilbert functions of artinian Gorenstein algebras presented by quadrics.

Question 2.10. *Is it true that for an artinian Gorenstein algebra presented by quadrics, with socle degree $e = r - i$, that the Hilbert function is uniquely determined, and that in particular $h_2 = \binom{r}{2} - \binom{i+2}{2} + 1$?*

We now show that this question does not have an affirmative answer, first by a simple example (which further illustrates the use of Lemma 2.5) and then in a more methodical manner. In the next section, we will show that nevertheless, it is the case in some instances.

Example 2.11. We know that if $r \geq 2$, then $(1, r, 1)$ and $(1, r, r, 1)$ are h -vectors of Gorenstein algebras presented by quadrics. Applying this, we get:

Socle degree 4: For all integers $s, t \geq 2$, the vector $(1, s + t, st + 2, s + t, 1)$ is the h -vector of a Gorenstein algebra presented by quadrics. Indeed, this follows by applying Proposition 2.4 to the Gorenstein vectors $(1, s, 1)$ and $(1, t, 1)$.

Note that applying this with $s = t = 4$ we get the h -vector $(1, 8, 18, 8, 1)$, which is not in the lists given in Example 2.9. Similarly, one can easily check that $(1, 6, 10, 6, 1)$ and $(1, 6, 11, 6, 1)$ are Gorenstein h -vectors.

Socle degree 5: For all integers s, t with $s \geq 2$ and $t \geq 3$, the vector $(1, s + t, st + t + 1, st + t + 1, s + t, 1)$ is the h -vector of a Gorenstein algebra presented by quadrics. Indeed, this follows by applying Proposition 2.4 to the Gorenstein vectors $(1, s, 1)$ and $(1, t, t, 1)$. The latter one exists by Corollary 2.7.

We do not know the answer to the following question:

Question 2.12. Are the h -vectors listed in Example 2.11 *all* the h -vectors of Gorenstein algebras presented by quadrics with socle degree 4 and 5, respectively? What about the analogous question for higher socle degree?

For fixed r , clearly there is a unique h -vector (for artinian Gorenstein algebras of any kind) when $e = 1, 2, 3$. We now show that for all socle degrees $4 \leq e \leq r - 2$ (hence assuming $r \geq 6$), Question 2.10 has a negative answer. The next section explores what happens when $e = r - 1$. An interesting extension of the case $e = 3$ is also explored in the next section.

Proposition 2.13. *Fix an integer $e \geq 4$. Then for any $r \geq e + 2$, there are at least two artinian Gorenstein algebras presented by quadrics, with $h_1 = r$ and having different values of h_2 .*

Proof. We will use the result of Example 2.9, and in particular the fact that for socle degree $e = r - i$ there exists an artinian Gorenstein algebra presented by quadrics, with $h_2 = \binom{r}{2} - \binom{i+2}{2} + 1$. In our situation now we assume $4 \leq e \leq r - 2$.

First say $e - 2 = (r - 2) - j$, with $2 \leq j \leq r - 4$. We know that there is an artinian Gorenstein algebra presented by quadrics with socle degree 2 and h -vector $(1, 2, 1)$, and one with socle degree $e - 2$ and h -vector $(1, r - 2, h_2, \dots)$, where $h_2 = \binom{r-2}{2} - \binom{j+2}{2} + 1$. Proposition 2.4 then implies that there is an artinian Gorenstein algebra presented by quadrics with socle degree e and codimension r , whose Hilbert function in degree 2 has value

$$\binom{r-2}{2} - \binom{j+2}{2} + 2r - 2.$$

Now, with the same value of j as above, we write $e - 2 = (r - 3) - (j - 1)$. We know that there is an artinian Gorenstein algebra presented by quadrics with socle degree 2 and h -vector $(1, 3, 1)$, and one with socle degree $e - 2$ and h -vector $(1, r - 3, h'_2, \dots)$, where $h'_2 = \binom{r-3}{2} - \binom{j+1}{2} + 1$. Again invoking Proposition 2.4, we obtain an artinian Gorenstein algebra presented by quadrics with socle degree e and codimension r , whose Hilbert function in degree 2 has value

$$\binom{r-3}{2} - \binom{j+1}{2} + 3r - 7.$$

We have to show that these are different values in degree 2. Suppose otherwise. A simple computation gives that then $j = 1$. Contradiction. \square

3. SOCLE DEGREES $r - 1$ AND 3

We have seen in Proposition 2.13 that whenever $4 \leq e \leq r - 2$, more than one Hilbert function is possible among artinian Gorenstein algebras presented by quadrics. We also know that in any case we have $e \leq r$, with equality if and only if R/I is a complete intersection. We thus can ask what happens when $e = r - 1$. Furthermore, there is one interesting question related to the case $e = 3$ that we have not yet addressed: if $h_2 = h_1 = r$, does this force $e = 3$? Clearly for these questions we obtain a stronger result by assuming that we are in the larger class of Gorenstein artinian algebras containing a regular sequence of r quadrics.

Notice that for $r = 3$ and $r = 4$, the only possible Hilbert functions with $e = r - 1$, containing a regular sequence of r quadrics, are $(1, 3, 1)$ and $(1, 4, 4, 1)$ respectively. Thus we lose nothing in assuming $r \geq 5$ in the following result.

Theorem 3.1. *Assume that $r \geq 5$ and that R/I is an artinian Gorenstein algebra, where $R = k[x_1, \dots, x_r]$ and I contains a regular sequence of r quadrics and no linear form. Assume that the socle degree of R/I is $e = r - 1$ and denote the h -vector of R/I by $(1, r, h_2, \dots, h_2, r, 1)$. Then*

- (a) h_2 must be either $\binom{r}{2} - 2$, $\binom{r}{2} - 1$ or $\binom{r}{2}$, and all of possibilities do occur.
- (b) If R/I is presented by quadrics then $h_2 = \binom{r}{2} - 2$.
- (c) If $h_2 = \binom{r}{2} - 2$ then R/I is presented by quadrics, and the entire Hilbert function of R/I is uniquely determined, namely it is

$$h_j = \binom{r-1}{j} + \binom{r-3}{j-1}$$

(and hence is one of the Hilbert functions given in Group 1 in Example 2.10).

- (d) For $r \geq 7$, if $h_2 = \binom{r}{2} - 1$ or $\binom{r}{2}$ then the Hilbert function of R/I is not uniquely determined, at least if the base field k has characteristic zero.

Proof. We will use the formula for the behavior of the Hilbert function under linkage (cf. [6] Theorem 3, [15] Corollary 5.2.19, [16] Corollary 9).

We know that $h_2 \leq \binom{r}{2}$. Let $h_2 = \binom{r}{2} - \alpha$, where $\alpha \geq 0$. A complete intersection of r quadrics links I to an almost complete intersection ideal J , and after a short calculation we see that the Hilbert function of R/J is

$$\left(1, r-1, \binom{r-1}{2} - 1, \binom{r-1}{3} - (r-1) + \alpha, \dots, \binom{r}{2} - h_3, \alpha\right)$$

where the last α (possibly 0) occurs in degree $r - 2$.

Viewing J as an ideal in a ring R' with $r - 1$ variables, we can link using a complete intersection of $r - 1$ quadrics, obtaining as the residual a Gorenstein ideal I' with Hilbert function

$$(3.1) \quad \left(1, r-1-\alpha, h_3 - \binom{r-1}{3}, \dots, r-1-\alpha, 1\right)$$

and socle degree $r - 3$.

If $\alpha = 2$, we may view I' as being in a ring R'' with $r - 3$ variables. Since it is a quotient of a complete intersection of quadrics, with the same socle degree as the complete intersection, I' must itself be a complete intersection of quadrics. Thus the Hilbert function of R''/I' is uniquely determined. It is $\dim_k[R''/I']_j = \binom{r-3}{j}$. By linkage, this determines the Hilbert function of R/I , proving the second part of (c).

Now we follow the resolutions of these linked ideals. For simplicity, we will now view all ideals as being in R , so the minimal free resolution of R/I' has the form

$$\begin{array}{ccccccc} & & R(-2)^3 & & & & \\ & & \oplus & & R(-1)^3 & & \\ 0 \rightarrow & R(-2r+3) \rightarrow \dots \rightarrow & R(-3)^{3r-9} & \rightarrow & \oplus & \rightarrow R \rightarrow R/I' \rightarrow 0. \\ & & \oplus & & R(-2)^{r-3} & & \\ & & R(-4)^{\binom{r-3}{2}} & & & & \end{array}$$

Now J is obtained from I' by linking using a regular sequence of one linear form and $r - 1$ quadrics in R , so by a standard mapping cone computation (splitting off one copy

of $R(-1)$ and $r - 3$ copies of $R(-2)$), the minimal free resolution of R/J has the form

$$0 \rightarrow R(2 - 2r)^2 \rightarrow \begin{array}{c} R(3 - 2r)^5 \\ \oplus \\ R(4 - 2r)^{2r-6} \end{array} \rightarrow \cdots \rightarrow \begin{array}{c} R(-1) \\ \oplus \\ R(-2)^r \end{array} \rightarrow R \rightarrow R/J \rightarrow 0$$

(all syzygies are Koszul; notice that the exponent of $R(4 - 2r)$ is $2r - 6$ and not $2r - 8$). Finally, we obtain I from J by linking using a regular sequence of r quadrics, obtaining the free resolution

$$\cdots \rightarrow R(-2)^{r+2} \rightarrow R \rightarrow R/I \rightarrow 0$$

(and no splitting of any copies of $R(-2)$ is possible). Thus R/I is presented by $r + 2$ quadrics. This completes the proof of (c).

If $\alpha \geq 3$ then the socle degree of R''/I' is greater than the number of variables, which is impossible for a quotient of a complete intersection of quadrics. This proves the first half of (a). Then (b) follows immediately: if $\alpha = 0$ then R/I is a complete intersection, so the socle degree must be r rather than $r - 1$, while if $\alpha = 1$ then it violates Proposition 2.3(5).

Next we prove the existence of R/I for $e = r - 1$ ($r \geq 5$) and the three possible resulting values of h_2 . This follows from the links described above. Indeed, from Corollary 2.8 we know that there exist Gorenstein algebras with socle degree $r - 3$ in $r - 1, r - 2$ and $r - 3$ variables, respectively. So suppose that R/I' is such an algebra, with Hilbert function $(1, r - \beta, g_2, \dots, g_2, r - \beta, 1)$ with $\beta = 1, 2$ or 3 . In all three cases we will view I' as being artinian in a ring R' in $r - 1$ variables, so if $\beta = 2$ or 3 then we view I' as also containing 1 or 2 linear forms, respectively. If we link with a regular sequence of quadrics in R' , the residual is an almost complete intersection, J , with Hilbert function

$$\left(1, r - 1, \binom{r - 1}{2} - 1, \dots, \binom{r - 1}{3} - g_3, \binom{r - 1}{2} - g_2, \beta - 1\right)$$

where the $\beta - 1$ occurs in degree $r - 2$. Now view J as being artinian in a ring, R , with r variables. Notice that J contains r minimal generators of degree 2. Thus we may link J in R using a regular sequence consisting of r quadrics, all minimal generators of J . The residual is a Gorenstein ideal containing a regular sequence of quadrics. Its Hilbert function has the form

$$\left(1, r, \binom{r}{2} - \beta + 1, \dots, r, 1\right)$$

with socle degree $r - 1$. Since β ranges from 1 to 3, we are done with (a).

Finally, we have to prove (d), namely the non-uniqueness of the Hilbert function of R/I when $h_2 = \binom{r}{2} - 1$ and $h_2 = \binom{r}{2}$. First we will consider the case $h_2 = \binom{r}{2} - 1$. Our proof will be by induction on r .

First, assume that r is odd. As a building block, notice that both $(1, 2, 1)$ and $(1, 3, 1)$ are the Hilbert functions of artinian Gorenstein algebras presented by quadrics. By adding linear forms to the ideal, we can view both of these as being quotients of ideals in 4 variables. Linking with complete intersections of quadrics, we obtain residuals that are almost complete intersections, with respective Hilbert functions

$$(1, 4, 5, 2) \quad \text{and} \quad (1, 4, 5, 1).$$

In a similar way, we can link in a ring with 5 variables, and we can arrange that the five quadrics achieving the link are minimal generators of the ideal. Hence the residual is Gorenstein in both cases, with Hilbert functions, respectively,

$$(1, 5, 8, 5, 1) \quad \text{and} \quad (1, 5, 9, 5, 1).$$

Now link twice, again, this time in a ring with 6 variables and then in a ring with 7 variables. We obtain Gorenstein algebras with Hilbert functions, respectively,

$$(1, 7, 20, 28, 20, 7, 1) \quad \text{and} \quad (1, 7, 20, 29, 20, 7, 1).$$

Notice that $20 = \binom{7}{2} - 1$. This begins the induction. Now suppose that R''/I_{r-2} and R''/I'_{r-2} are Gorenstein algebras with socle degree $r - 3$ (r odd) and Hilbert functions both of the form

$$(1, r - 2, \binom{r - 2}{2} - 1, \dots, \binom{r - 2}{2} - 1, r - 2, 1),$$

which are the same except in the middle degree, where they differ by 1. Here R'' is a polynomial ring with $r - 2$ variables. If we view these ideals in a ring with $r - 1$ variables and link with a complete intersection of quadrics, we obtain almost complete intersections with Hilbert functions

$$(1, r - 1, \binom{r - 1}{2} - 1, \dots, \gamma, \dots, \binom{r - 1}{2} - \binom{r - 2}{2} + 1, 1)$$

where the socle degree occurs in degree $r - 2$ and the values are the same in all degrees except one, where the values of γ differ by 1. Linking again inside a ring R of r variables, we obtain Gorenstein algebras with Hilbert functions

$$(1, r, \binom{r}{2} - 1, \dots, \binom{r}{2} - \binom{r - 1}{2} + 1, 1)$$

(notice that $\binom{r}{2} - \binom{r - 1}{2} + 1 = r$), with socle degree $r - 1$ which agree in all degrees but one. In fact, by the symmetry of the Hilbert function, we can even deduce that this one degree must be in the middle, without making the computation. This completes the case where r is odd. When r is even, the same construction works if we begin with the building blocks $(1, 3, 3, 1)$ and $(1, 4, 4, 1)$. We first link inside a ring with 5 variables, and then inside a ring with 6 variables, obtaining Gorenstein Hilbert functions

$$(1, 6, 13, 13, 6, 1) \quad \text{and} \quad (1, 6, 14, 14, 6, 1).$$

Repeating the procedure, we obtain Gorenstein Hilbert functions

$$(1, 8, 27, 48, 48, 27, 8, 1) \quad \text{and} \quad (1, 8, 27, 49, 49, 27, 8, 1).$$

This starts the induction, which then proceeds as before.

Finally, we prove that the Hilbert function is not unique when $h_2 = \binom{r}{2}$. First notice that if $r = 5$ or 6 then the condition that $h_2 = \binom{r}{2}$ does force the Hilbert function to be unique, so the assumption that $r \geq 7$ is necessary. Next, notice that if R/\mathfrak{a} is an artinian Gorenstein algebra with socle degree e , and if L is a linear form, then $R/(\mathfrak{a} : L)$ is an artinian Gorenstein algebra with socle degree $e - 1$. Hence taking \mathfrak{a} to be the ideal (x_1^2, \dots, x_r^2) , we can produce artinian Gorenstein algebras of socle degree $r - 1$.

Let $L = x_1 + \dots + x_5$. We claim that (i) the multiplication

$$\times L : [R/\mathfrak{a}]_2 \rightarrow [R/\mathfrak{a}]_3$$

is injective (hence $h_2 = \binom{r}{2}$ for $R/(\mathfrak{a} : L)$) and that (ii) the corresponding multiplication from degree 3 to degree 4 is not injective, so $h_3 < \binom{r}{3}$. Since it is known that this multiplication is injective for general L ([22], [24], [21]) (and in fact even for $L = x_1 + \dots + x_r$ - cf. [18]), the non-uniqueness follows. (Here we are using the assumption that the characteristic is zero.)

For the proof of (ii), it is enough to observe that $x_1x_2x_4 - x_2x_3x_4 - x_1x_4x_5 + x_3x_4x_5$ is in the kernel of $\times L$. Finally, we prove (i). By the duality of $R/(x_1^2, \dots, x_r^2)$, it is equivalent to prove that

$$\times L : [R/\mathfrak{a}]_{r-3} \rightarrow [R/\mathfrak{a}]_{r-2}$$

is surjective. Since the cokernel of this map is isomorphic to $[R/(\mathfrak{a} + (L))]_{r-2}$, we have to show that this latter is zero. Equivalently, we must show that

$$[k[x_2, \dots, x_r]/(x_2^2, \dots, x_r^2, (-x_2 - x_3 - x_4 - x_5)^2)]_{r-2} = 0.$$

That is, we must show that

$$[k[x_2, \dots, x_r]/(x_2^2, \dots, x_r^2, x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5)]_{r-2} = 0.$$

To do this, it is enough to show that the square-free monomials of degree $r - 2$ in $k[x_2, \dots, x_r]$ reduce to zero. But this is clear; for instance,

$$x_2 \cdots x_{r-1} = (x_2x_3 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5)(x_4 \cdots x_{r-1})$$

in $k[x_2, \dots, x_r]/(x_2^2, \dots, x_r^2)$. \square

Example 3.2. Following the construction given in the previous proof, we can begin with a R a polynomial ring in 5 variables and take a Gorenstein algebra R/I_2 with h -vector $(1, 3, 1)$ (so I_2 contains two linear forms). Then linking successively with complete intersections of type $(1, 2, 2, 2, 2)$ and $(2, 2, 2, 2, 2)$, we obtain a Gorenstein algebra with Hilbert function $(1, 5, 9, 5, 1)$. Thus $e = r - 1 = 4$ and $h_2 = \binom{r}{2} - 1 = 9$. The minimal free resolution of R/I is

$$0 \rightarrow R(-9) \rightarrow \begin{array}{c} R(-6) \\ \oplus \\ R(-7)^6 \end{array} \rightarrow \begin{array}{c} R(-5)^2 1 \\ \oplus \\ R(-6) \end{array} \rightarrow \begin{array}{c} R(-3) \\ \oplus \\ R(-4)^2 1 \end{array} \rightarrow \begin{array}{c} R(-2)^6 \\ \oplus \\ R(-3) \end{array} \rightarrow R \rightarrow R/I \rightarrow 0.$$

Notice that this does not give a negative answer to Question 2.12 since I has a cubic generator.

The results thus far allow us to describe the possible Hilbert functions of artinian Gorenstein algebras presented by quadrics with small codimension.

Proposition 3.3. *For $r \leq 5$ the following are the only h -vectors for artinian Gorenstein algebras presented by quadrics.*

r	h -vectors
2	$(1, 2, 1)$
3	$(1, 3, 1), (1, 3, 3, 1)$
4	$(1, 4, 1), (1, 4, 4, 1), (1, 4, 6, 4, 1)$
5	$(1, 5, 1), (1, 5, 5, 1), (1, 5, 8, 5, 1), (1, 5, 10, 10, 5, 1)$

4. INJECTIVITY – CONJECTURES AND PARTIAL RESULTS

In this section we make two conjectures, and we prove them in some cases and explore the connections between them. In the next section we will see some consequences, propose another conjecture, and relate them to our first conjecture here.

Conjecture 4.1. (Injectivity Conjecture) *Let R/I be an artinian Gorenstein algebra presented by quadrics, of socle degree ≥ 3 , and assume that R is defined over a field of characteristic $\neq 2$. Let L be a general linear form. Then the multiplication $\times L : (R/I)_1 \rightarrow (R/I)_2$ is injective.*

Notice that the assumption on the characteristic is necessary, as shown by the example of the complete intersection $I = (x_1^2, \dots, x_r^2)$, where $L^2 \in I$ for every linear form L if the characteristic is two.

In the following we will use a result of Huneke and Ulrich, which we now recall. For a finitely generated graded module A , denote by $a_-(A)$ the initial degree of A .

Lemma 4.2 (Socle Lemma [12]). *Let M be a nonzero finitely generated graded module over a polynomial ring $R = k[x_1, \dots, x_r]$, where k is a field of characteristic zero. Set $\mathfrak{m} = (x_1, \dots, x_r)$. Let $x \in [R]_1$ be a general linear form, and let*

$$0 \rightarrow K \rightarrow M(-1) \xrightarrow{x} M \rightarrow C \rightarrow 0$$

be exact. If $K \neq 0$ then $a_-(K) > a_-(\text{soc}(C))$.

Proposition 4.3. *Assume that $\text{char } k = 0$. Then for any complete intersection $I = (Q_1, \dots, Q_r)$ of quadrics, the Injectivity Conjecture is true.*

Proof. Let L be a general linear form. Consider the exact sequence

$$0 \rightarrow (I : L)/I(-1) \rightarrow (R/I)(-1) \xrightarrow{\times L} R/I \rightarrow R/(I, L) \rightarrow 0.$$

From the Socle Lemma we obtain that if a is the least degree of an element in $I : L$ that is not in I , then a is greater than or equal to the least degree of a socle element of $R/(I, L)$ (notice the twist by -1). Suppose that $\times L$ is not injective as claimed. From the above sequence in degree 2, we obtain that $(I : L)$ contains an element of degree 1, so $R/(I, L)$ has a socle element in degree 1. Furthermore, we obtain in this case that

$$\dim(R/(I, L))_2 \geq \binom{r}{2} - r + 1 = \binom{r-1}{2}.$$

Using the symmetry of the Hilbert function of the artinian Gorenstein algebra $R/(I : L)$, we get that $\dim(R/(I : L)(-1))_{r-1} < r$, so the sequence

$$0 \rightarrow R/(I : L)(-1) \rightarrow R/I \rightarrow R/(I, L) \rightarrow 0$$

in degree $r - 1$ now gives $\dim(R/(I, L))_{r-1} \neq 0$.

Now, in the ring $R/(L)$, the artinian ideal $(I, L)/(L)$ is generated by quadrics, its Hilbert function is $\geq \binom{r-1}{2}$ in degree 2, and it has socle degree $r - 1$. Thus it must be the complete intersection of $r - 1$ quadrics. But then the socle element in degree 1 gives a contradiction. \square

Corollary 4.4. *Any complete intersection of ≤ 4 quadrics has the WLP.*

In characteristic zero we form a stronger conjecture (although we do not have a counterexample in positive characteristic).

Conjecture 4.5. (WLP Conjecture) *Let R/I be an artinian Gorenstein algebra presented by quadrics, of socle degree ≥ 3 , and assume that R is defined over a field of characteristic zero. Then R/I has the WLP.*

Remark 4.6. In this conjecture the Gorenstein assumption can not be dropped. For example, if the ideal I is generated by squares of $r + 1$ general linear forms in an even number of variables, r , then R/I does not have the WLP by Theorem 5.1 in [19].

We are able to give a partial result toward the WLP Conjecture, in the important special case of a complete intersection of quadrics.

Corollary 4.7. *Let R/I be an artinian complete intersection of quadrics, where $R = k[x_1, \dots, x_r]$ and $r \geq 5$, and assume that $\text{char } k = 0$. If L is a general linear form then multiplication by L from degree 2 to degree 3 has at most a 1-dimensional kernel.*

Proof. By Proposition 4.3, $I : L$ contains no linear forms, and it clearly contains a regular sequence of r quadrics. Notice that $R/(I : L)$ is Gorenstein with socle degree $r - 1$. Theorem 3.1 shows that the value of the Hilbert function of $R/(I : L)$ in degree 2 is between $\binom{r}{2} - 2$ and $\binom{r}{2}$. This means that the kernel of this multiplication is at most 2-dimensional.

Now suppose that this kernel is exactly 2-dimensional. Then the value of the Hilbert function of $R/(I : L)$ is $\binom{r}{2} - 2$. By Theorem 3.1, $R/(I : L)$ is presented by quadrics. Linking $(I : L)$ to (I, L) , a mapping cone argument as in Theorem 3.1 gives that $R/(I, L)$ is a level algebra of type 2, i.e. its socle is 2-dimensional and concentrated at the end, namely in degree $r - 2$ (where r is the socle degree of R/I). On the other hand, again applying the Socle Lemma, we see that the socle of $R/(I, L)$ must begin in degree ≤ 2 . Since $r \geq 5$, we have a contradiction. \square

Lemma 4.8. *Let R/I be an artinian Gorenstein algebra presented by quadrics. Assume that the Injectivity Conjecture is true for R/I . Assume that the value of the Hilbert function of R/I in degree 2 is h . Let Q_1, \dots, Q_{h-1} be a general set of quadrics, and consider the ideal $J_1 = (I, Q_1, \dots, Q_{h-1})$. Then R/J_1 is Gorenstein with Hilbert function $(1, r, 1)$.*

Proof. It is clear that the Hilbert function of R/J_1 begins $(1, r, 1, \dots)$. If the degree 3 entry is not 0 then it can only be 1, by Macaulay's growth condition. But then by Gotzmann persistence, J_1 cannot be generated by quadrics. Hence the Hilbert function assertion is proven.

We will see that the Gorenstein property relies on injectivity of multiplication by a general linear form on R/I from degree 1 to degree 2. Let L be any linear form. Let V be the vector space of forms of degree 2. We have to show that R/J_1 does not have a socle element in degree 1, i.e. that it is never the case that $L \cdot R_1 \subset \langle (I)_2, Q_1, \dots, Q_{h-1} \rangle$ in V (when Q_1, \dots, Q_{h-1} are chosen generically). Let

$$d = \dim(\ker[(R/I)_1 \xrightarrow{\times L} (R/I)_2]).$$

Note that

$$(4.1) \quad d = 0 \text{ if and only if } \times L \text{ is injective.}$$

Since in V we have that d is the dimension of the intersection of $L \cdot R_1$ and $(I)_2$, it follows that

$$(4.2) \quad \dim(\text{span}(L \cdot R_1, (I)_2)) = r + \binom{r+1}{2} - h - d.$$

In particular, by injectivity, we get for a general linear form L that

$$(4.3) \quad \dim(\text{span}(L \cdot R_1, (I)_2)) = r + \binom{r+1}{2} - h.$$

Notice that $(J_1)_2$ is a hyperplane in V . We now consider the projective space $\mathbb{P}V = \mathbb{P}^{\binom{r+1}{2}-1}$. Let

$$\begin{aligned} \Lambda &= \{\text{hyperplanes in } \mathbb{P}^{\binom{r+1}{2}-1} \text{ containing } \mathbb{P}I_2\} \\ \Sigma &= \{\text{linear subvarieties } \mathbb{P}(L \cdot R_1)\} \text{ for linear forms } L \\ \mathbb{I} &= \{(A, H) \in \Sigma \times \Lambda \mid A \subset H\} \end{aligned}$$

We have the projections

$$\begin{array}{ccc} \mathbb{I} & \subset & \Sigma \times \Lambda \\ \phi_1 \swarrow & & \searrow \phi_2 \\ \Sigma & & \Lambda \end{array}$$

What is the dimension of the generic fibre of ϕ_1 in \mathbb{I} ? By (4.3), we want to know how many hyperplanes in $\mathbb{P}^{\binom{r+1}{2}-1}$ contain a linear space of dimension $r + \binom{r+1}{2} - h - 1$ (since we seek the generic fibre, so L is general). Thus the generic fibre has dimension

$$\binom{r+1}{2} - 1 - \left[r + \binom{r+1}{2} - h - 1 \right] - 1 = h - r - 1.$$

Thus we obtain

$$\dim \mathbb{I} = \dim \Sigma + (h - r - 1) = (r - 1) + (h - r - 1) = h - 2.$$

On the other hand, $\dim \Lambda = h - 1$. Thus the image of \mathbb{I} under ϕ_2 cannot be dense in Λ . In other words, for a general choice of Q_1, \dots, Q_{h-1} , the ring $R/(I, Q_1, \dots, Q_{h-1})$ does not have socle in degree 1, and we are done. \square

Corollary 4.9. *Assume that R/I is presented by quadrics, and has socle degree $e = 4$ and h -vector $(1, r, h_2, r, 1)$. Assume that the Injectivity Conjecture holds for R/I . Then $h_2 \leq \left\lfloor \frac{r^2+2}{3} \right\rfloor$.*

Proof. Let Q_1, \dots, Q_{h_2-1} be generically chosen quadrics in R_2 . Let $J_1 = I + (Q_1, \dots, Q_{h_2-1})$. We saw in Lemma 4.8 that J_1 is Gorenstein. Let $K = I : J_1$. Using results for Hilbert functions and free resolutions under linkage, we conclude that R/K has h -vector $(1, r, h_2 - 1)$ and that K is generated by quadrics. This implies

$$r \cdot (\dim[K]_2) = r \cdot [\dim[R]_2 - h_2 + 1] \geq \dim[R]_3.$$

The claimed upper bound for h_2 follows with a simple calculation. \square

5. THE CASE $h_2 = r$

We have seen thus far that when $4 \leq e \leq r - 2$, there is not a unique Hilbert function for artinian Gorenstein algebras presented by quadrics and having socle degree e . On the other hand, when $e = r - 1$, the Hilbert function is unique. It remains to examine what happens when $e = 3$. On one hand this is trivial: if $e = 3$ then clearly (by symmetry), the only possible Hilbert function is $(1, r, r, 1)$. What is interesting for us now is the converse: if $h_2 = r$ then we would like to show that $e = 3$. Our final conjecture gives an even more general statement, in that it does not assume at first that the algebra is presented by quadrics. We assume that $r \geq 3$ since in codimension two the only Gorenstein algebras generated by quadrics are complete intersections of two quadrics, whose h -vectors are $(1, 2, 1)$.

Conjecture 5.1. (“ $h_2 = r$ ” Conjecture) *Let R/I be an artinian Gorenstein algebra of codimension $r \geq 3$ and socle degree e , and assume that $h_2 = r$. Then $h_i = r$ for all $i = 1, 2, \dots, e - 1$. Furthermore, if $e \geq 4$ then $(I)_{\leq e-1}$, the ideal generated by all forms in I whose degree is at most $e - 1$, is the saturated ideal of a zero-dimensional scheme, so I has an additional $r - 1$ minimal generators in degree e .*

We now show that if we have the injectivity from degree 1 to degree 2 (whether or not the algebra is presented by quadrics) then most of the “ $h_2 = r$ ” Conjecture is true.

Proposition 5.2. *Assume that R/I is an artinian Gorenstein algebra for which the multiplication by a general linear form on R/I from degree 1 to degree 2 is injective. Assume that $h_2 = r$, with socle degree $e \geq 3$. Then $h_i = r$ for all $i = 1, 2, \dots, e-1$. Furthermore, if R/I is presented by quadrics then $e = 3$.*

Proof. Let L be a general linear form, and let $J = (I, L)$. Let $J' = I : L$. Note that J' is a Gorenstein algebra, and R/J' has socle degree $e-1$. Write the h -vector of R/J' as $(1, b_1, b_2, \dots, b_2, b_1, 1)$. As noted earlier, I provides a G-link of J with J'

Suppose that $e \geq 4$. We are assuming that $b_1 = r$. Hence R/J' has h -vector $(1 \ r \ b_2 \ \dots \ b_2 \ r \ 1)$. We have the h -vector computation

deg	0	1	2	3	...	$e-2$	$e-1$	e
R/I	1	r	r	h_3	...	r	r	1
$(R/J')(-1)$	0	1	r	b_2	...	b_2	r	1
R/J	1	$r-1$	0	0	...	0	0	0

From the values in degrees $e-2$ and 3 we see that $r = b_2 = h_3$. (Note that this is precisely where we use the hypothesis that $e \geq 4$.) But then we obtain, using the values in degrees $e-3$ and 4, that $r = h_3 = b_3 = h_4$, and continuing in this manner we obtain the assertion about h_i .

It remains to show that if R/I is presented by quadrics then $e = 3$. The minimal free resolution of $J = (I, L)$ is easily computed. In particular, the resolution begins

$$\cdots \rightarrow R(-3)^N \rightarrow \begin{array}{c} R(-1) \\ \oplus \\ R(-2)^M \end{array} \rightarrow J \rightarrow 0$$

where $M = \binom{r}{2}$ and $N = \binom{r+1}{2} + r \cdot \binom{r}{2} - \binom{r+2}{3}$. On the other hand, from the h -vector it is easy to see that I has $\binom{r}{2}$ independent quadrics, and by hypothesis these generate the ideal. Since (I, L) has exactly $\binom{r}{2}$ quadrics as minimal generators, L is general, and $I \subset (I, L) = J$, we may assume that the minimal generators of I are minimal generators of J . Now, since I links J to J' , we can obtain a free resolution for J' from those of I and J . In particular, from our observation about the generators of I and about the resolution of J , we see that all copies of $R(-2)$ in the resolution of I split with the copies of $R(-2)$ in the resolution of J . Hence the end of the resolution of J' is

$$0 \rightarrow R(-e-r+1) \rightarrow R(-e-r+3)^N \rightarrow \dots$$

Note that this is not necessarily minimal, since copies of $R(-e-r+3)$ may have split in the mapping cone. In any case, since J' is Gorenstein, the self-duality of the resolution implies that J' is generated by quadrics.

Now, we know that $I \subset I : L$ but they are not equal, since their quotients have different socle degrees. Our observation that $h_2 = b_2 = r$ implies that $I_2 = (I : L)_2$. If both ideals were generated by quadrics, we would have $I = (I : L)$, which is a contradiction. This completes the proof. \square

Corollary 5.3. *Assume that $r \geq 3$ and assume that the Injectivity Conjecture is true. Then the following are equivalent for an artinian Gorenstein algebra R/I presented by quadrics, with Hilbert function $\underline{h} = \{h_i, i \geq 0\}$, and such an algebra exists:*

- (1) $h_2 = r$
- (2) $e = 3$
- (3) $\underline{h} = (1, r, r, 1)$.

For the remainder of this section, we consider the situation where R/I is presented by quadrics and $h_2 = r$. Our goal is to prove the result of Corollary 5.3 without assuming the Injectivity Conjecture. Thus without loss of generality we assume that multiplication on R/I from degree 1 to degree 2 is not injective. Equivalently, and using duality, we assume that $[R/(I, L)]_{e-1} \neq 0$ but that $[R/(I, L)]_e = 0$. We consider the generic initial ideal of I , $\text{gin}(I)$, with respect to the reverse lexicographic order. Throughout the remainder of this section we assume that our base field k has characteristic zero. Then the generic initial ideal is strongly stable.

Recall the result of Hoa and Trung, which we cite from Lemma 2.14 of [1]:

Lemma 5.4. *Let I be a homogeneous ideal in R and let $n = \dim(R/I)$. Then the s -reduction number $r_s(R/I)$ for $s \geq n$ can be given as the following:*

$$\begin{aligned} r_s(R/I) &= \min\{k \mid x_{n-s}^{k+1} \in \text{gin}(I)\} \\ &= \min\{k \mid h_{R/(I, J)}(k+1) = 0\} \end{aligned}$$

where J is an ideal generated by s general linear forms. Furthermore,

$$r_s(R/I) = r_s(R/\text{gin}(I)).$$

We also have the following from [2], Proposition 3.9:

Lemma 5.5. *Let $A = R/I$ be a graded artinian algebra with Hilbert function $\mathbf{H} = (1, r, \dots, h_s)$. Let $\mathcal{G}(\text{gin}(I))_i$ denote the set of minimal monomial generators of $\text{gin}(I)$ in degree i . If $d \geq r_1(A)$ then*

$$|\{T \in \mathcal{G}(\text{gin}(I))_{d+1} \mid x_r \text{ divides } T\}| = h_d - h_{d+1}.$$

In our situation we deduce the following:

- $n = \dim(R/I) = 0$ and $r_1(R/I) = r_1(R/\text{gin}(I)) = e - 1$.
- $x_{r-1}^e \in \text{gin } I$, but $x_{r-1}^{e-1} \notin \text{gin } I$.
- $\text{gin}(I)$ has $r - 1$ minimal generators of degree e that are divisible by x_r .

Since $h_{R/I}(e) = 1$, $\text{gin}(I)$ contains all the monomials of degree e except x_r^e . Since $h_{R/I}(e - 1) = r$, we can use the above information to compute the monomials that are *not* in $\text{gin}(I)$, and at the same time consider which monomials of degree e in $\text{gin}(I)$ have to be minimal generators. Note that there are exactly r monomials of degree $e - 1$ that are not in $\text{gin}(I)$. We have seen that $x_{r-1}^{e-1} \notin \text{gin}(I)$.

Lemma 5.6. *None of the monomials $x_{r-1}^i x_r^{e-1-i}$ ($0 \leq i \leq e - 1$) are in $\text{gin}(I)$. There are e such monomials.*

Proof. If any of these were in $\text{gin}(I)$ then by the Borel property $x_{r-1}^{e-1} \in \text{gin}(I)$, which we have seen is not the case. \square

Corollary 5.7. *The monomials $x_{r-1}^i x_r^{e-i}$ ($1 \leq i \leq e$) are all minimal generators of $\text{gin}(I)$. There are e such monomials, but only $e - 1$ that are divisible by x_r .*

Proof. If not, then either $x_{r-1}^{i-1} x_r^{e-i}$ or $x_{r-1}^i x_r^{e-i-1} \in \text{gin}(I)$. Either way, $x_{r-1}^{e-1} \in \text{gin}(I)$ by the Borel property, again contradicting our observation above. Note that $x_r^e \notin \text{gin}(I)$, although $x_r^{e+1} \in \text{gin}(I)$. \square

Lemma 5.8. *The monomial $x_{r-2} x_r^{e-2} \notin \text{gin}(I)$, provided $e \geq 4$.*

Proof. Suppose it were. Then by the Borel property, the only monomials of degree $e - 1$ that might not be contained in $\text{gin}(I)$ are those of the form $x_{r-1}^i x_r^{e-1-i}$ (as seen above). But this is not enough to give $h_{R/I}(e - 1) = r$. \square

Lemma 5.9. *If the monomial $x_{r-2}x_{r-1}x_r^{e-3}$ is in $\text{gin}(I)$ then the only monomials of degree $e - 1$ that might not be contained in $\text{gin}(I)$ are those of the form $x_{r-1}^i x_r^{e-1-i}$ ($0 \leq i \leq e - 1$ as seen above), and $x_j x_r^{e-2}$ ($1 \leq j \leq r - 2$). These total $r + e - 2$.*

Proof. It follows by the Borel property and the above considerations. \square

We need the following result that is probably known to experts.

Proposition 5.10. *Let I be an artinian ideal and let L be a general linear form. Then for any integer d , the rank of the homomorphism $(\times L) : [R/I]_d \rightarrow [R/I]_{d+1}$ is the same as the rank of the homomorphism $(\cdot x_r) : [R/\text{gin}(I)]_d \rightarrow [R/\text{gin}(I)]_{d+1}$.*

Proof. We have the exact sequences

$$(5.1) \quad 0 \rightarrow \left[\frac{\text{gin}(I) : x_r}{\text{gin}(I)} \right]_d \rightarrow [R/\text{gin}(I)]_d \xrightarrow{\cdot x_r} [R/\text{gin}(I)]_{d+1} \rightarrow [R/(\text{gin}(I), x_r)]_{d+1} \rightarrow 0$$

and

$$0 \rightarrow \left[\frac{I : L}{I} \right]_d \rightarrow [R/I]_d \xrightarrow{\cdot L} [R/I]_{d+1} \rightarrow [R/(I, L)]_{d+1} \rightarrow 0.$$

The dimension of the last term determines the rank of the multiplication. But by [10],

$$(5.2) \quad \text{gin} \left(\frac{(I, L)}{(L)} \right) = \frac{(\text{gin}(I), x_r)}{(x_r)},$$

where the ideal $(I, L)/(L)$ is considered as an ideal in $k[x_1, \dots, x_{r-1}]$. This implies that $\dim[R/(\text{gin}(I), x_r)]_{d+1} = \dim[R/(I, L)]_{d+1}$. \square

Proposition 5.11. *Write $h_{R/I}(t) = h_t$. (Here I is not necessarily Gorenstein.) Consider the homomorphism $(\times L) : [R/I]_d \rightarrow [R/I]_{d+1}$. Then*

(1) *The rank of this homomorphism is*

$$\text{rk}(\times L) = \left(\begin{array}{l} \# \text{ of monomials of degree } d + 1 \text{ not in } \text{gin}(I) \\ \text{that are divisible by } x_r. \end{array} \right)$$

(2) *Assume that $h_d = h_{d+1}$. Then*

$$|\{T \in \mathcal{G}(\text{gin}(I))_{d+1} \mid x_r \text{ divides } T\}| = \left(\begin{array}{l} \# \text{ of monomials of degree } d + 1 \text{ not in } \text{gin}(I) \\ \text{and not divisible by } x_r. \end{array} \right)$$

Proof. It follows from the exact sequence (5.1) together with Proposition 5.10. \square

Example 5.12. Let R/I be an artinian Gorenstein algebra presented by quadrics, with $r = 10$ and $e = 6$. Let us show that the Hilbert function cannot be

$$1 \ 10 \ 10 \ * \ 10 \ 10 \ 1.$$

There are 10 monomials of degree 5 that are not in $\text{gin}(I)$, and among them are

$$x_9^5, x_9^4 x_{10}, x_9^3 x_{10}^2, x_9^2 x_{10}^3, x_9 x_{10}^4, x_{10}^5,$$

by Lemma 5.6. One can check that any other monomial *not divisible by* x_{10} must be in $\text{gin}(I)$. For instance, if $x_8 x_9^4 \notin \text{gin}(I)$ then also the product of x_8 with any degree 4 monomial in the variables x_9 and x_{10} must fail to be in $\text{gin}(I)$, giving a total of five, which together with the above list leaves too many not in $\text{gin}(I)$ (since $h_5 = 10$).

But now this means that there is only one monomial of degree 5 not in $\text{gin}(I)$ and not divisible by x_{10} . Hence the rank of multiplication by x_{10} , on $R/\text{gin}(I)$, from degree 4 to degree 5 is 9, and by duality this is also the rank of the multiplication from degree 1 to degree 2.

Let L be a general linear form. We get the usual diagram

deg	0	1	2	3	4	5	6
$h_{R/I}$	1	10	10	*	10	10	1
$h_{R/(I:L)(-1)}$	0	1	9	*	*	9	1
$h_{R/(I,L)}$	1	9	1	*	*	1	

This gives an immediate contradiction (see the proof below for details).

The following result is an application of these methods.

Theorem 5.13. *Let R/I be an artinian Gorenstein algebra presented by quadrics over $R = k[x_1, \dots, x_r]$ and having socle degree $e \geq 4$. Assume that $r < 4e - 6$ and that $h_2 = r$. Then multiplication by a general linear form on R/I from degree 1 to degree 2 must be an injection (hence isomorphism).*

Proof. The result is not hard to show for $r \leq 3$, so without loss of generality we can assume $r \geq 4$. Suppose that this multiplication is not an injection. Then by duality, the analogous homomorphism from degree $e - 2$ to degree $e - 1$ is not a surjection. However, the induced homomorphism from degree $e - 1$ to degree e is surjective, so we obtain $r_1(R/I) = e - 1$. Thus the results above apply.

We need to estimate the number of monomials of degree $e - 1$ not in $\text{gin}(I)$ and not divisible by x_r . First we focus on just estimating the number of monomials not in $\text{gin}(I)$. We know that

$$x_{r-1}^{e-1}, x_{r-1}^{e-2}x_r, \dots, x_r^{e-1}$$

are e such monomials. Note that only one of these is not divisible by x_r .

Suppose that M_1 is a monomial of degree $e - 1$ not in $\text{gin}(I)$ and not divisible by x_r , and not in the list above. Then also $x_{r-2}x_{r-1}^{e-2}$ will fail to be in $\text{gin}(I)$, and so

$$x_{r-2}x_{r-1}^{e-2}, x_{r-2}x_{r-1}^{e-3}x_r, \dots, x_{r-2}x_r^{e-2}$$

will fail to be in $\text{gin}(I)$. There are $e - 1$ such monomials, but only one is not divisible by x_r .

Suppose that in addition, there is another monomial, M_2 , not in $\text{gin}(I)$ and not divisible by x_r . Then at least one of the following possibilities must occur:

Case 1: $x_{r-2}^2x_{r-1}^{e-3} \notin \text{gin}(I)$. This forces any monomial of the form

$$x_{r-2}^2 \cdot (\text{monomial of degree } e - 3 \text{ in } x_{r-1} \text{ and } x_r)$$

to fail to be in $\text{gin}(I)$. There are $e - 2$ such monomials, but only one not divisible by x_r .

Case 2: $x_{r-3}x_{r-1}^{e-2} \notin \text{gin}(I)$. This forces any monomial of the form

$$x_{r-3} \cdot (\text{monomial of degree } e - 2 \text{ in } x_{r-1} \text{ and } x_r)$$

to fail to be in $\text{gin}(I)$. There are $e - 1$ such monomials, but only one not divisible by x_r .

We now investigate the consequences if there is another monomial, M_3 , not in $\text{gin}(I)$ and not divisible by x_r .

If we are in Case 1, then at least one of the following must occur:

- $x_{r-2}^3 x_{r-1}^{e-4} \notin \text{gin}(I)$. Then in addition to the monomials already listed, we get that

$$x_{r-2}^3 \cdot (\text{monomial of degree } e-4 \text{ in } x_{r-1} \text{ and } x_r)$$

all fail to be in $\text{gin}(I)$. There are $e-3$ such monomials.

- $x_{r-3} x_{r-1}^{e-2} \notin \text{gin}(I)$. Then in addition to the monomials already listed previously, we get that

$$x_{r-3} \cdot (\text{monomial of degree } e-2 \text{ in } x_{r-1} \text{ and } x_r)$$

all fail to be in $\text{gin}(I)$ (these were not listed yet in Case 1). There are $e-1$ such monomials, and only one is not divisible by x_r .

- $x_{r-3} x_{r-2} x_{r-1}^{e-3} \notin \text{gin}(I)$. Then we obtain that the following two sets:

$$x_{r-3} \cdot (\text{monomial of degree } e-2 \text{ in } x_{r-1} \text{ and } x_r)$$

and

$$x_{r-3} x_{r-2} \cdot (\text{monomial of degree } e-3 \text{ in } x_{r-1} x_r)$$

all must fail to be in $\text{gin}(I)$. There are $(e-1) + (e-2) = 2e-3$ such monomials, two of which are not divisible by x_r .

If we are in Case 2, then at least one of the following must occur:

- $x_{r-2}^2 x_{r-1}^{e-3} \notin \text{gin}(I)$. This forces any monomial of the form

$$x_{r-2}^2 \cdot (\text{monomial of degree } e-3 \text{ in } x_{r-1} \text{ and } x_r)$$

to fail to be in $\text{gin}(I)$. There are $e-2$ such monomials, but only one not divisible by x_r .

- $x_{r-3} x_{r-2} x_{r-1}^{e-3} \notin \text{gin}(I)$. Then we obtain that the following two sets:

$$x_{r-3} \cdot (\text{monomial of degree } e-2 \text{ in } x_{r-1} \text{ and } x_r)$$

and

$$x_{r-3} x_{r-2} \cdot (\text{monomial of degree } e-3 \text{ in } x_{r-1} x_r)$$

all must fail to be in $\text{gin}(I)$. There are $(e-1) + (e-2) = 2e-3$ such monomials, two of which are not divisible by x_r .

- $x_{r-4} x_{r-1}^{e-2} \notin \text{gin}(I)$. This forces any monomial of the form

$$x_{r-4} \cdot (\text{monomial of degree } e-2 \text{ in } x_{r-1} \text{ and } x_r)$$

to fail to be in $\text{gin}(I)$. There are $e-1$ such monomials, one of which is not divisible by x_r .

If M_1, M_2, M_3 all exist, we see that then there are at least $e + (e-1) + (e-2) + (e-3) = 4e-6$ monomials of degree $e-1$ not in $\text{gin}(I)$. Since $r < 4e-6$ by hypothesis, and $h_{e-1} = r$, this is a contradiction. Hence not all three monomials M_i exist. The most difficult situation is when both M_1 and M_2 exist, so we assume this.

We thus have that the only monomials of degree $e-1$ that can fail to be in $\text{gin}(I)$ and fail to be divisible by x_r are x_{r-1}^{e-1} , $x_{r-2} x_{r-1}^{e-2}$ and either $x_{r-2}^2 x_{r-1}^{e-3}$ or $x_{r-3} x_{r-1}^{e-2}$ (depending

on whether we are in Case 1 or Case 2, respectively). Then by Proposition 5.11, the rank of the multiplication $(\times L)$ from degree $e - 2$ to degree $e - 1$ is $r - 3$. (If M_2 also did not exist, it would be $r - 1$ or $r - 2$, and the proof would be similar though easier.)

As in Example 5.12 we obtain a diagram

deg	0	1	2	...	$e - 2$	$e - 1$	e
$h_{R/I}$	1	r	r	...	r	r	1
$h_{R/(I:L)(-1)}$	0	1	$r - 3$...	*	$r - 3$	1
$h_{R/(I,L)}$	1	$r - 1$	3	...	*	3	0

If $h_{R/(I,L)}(3) = 4$, then the Hilbert function has maximal growth from degree 2 to degree 3, and so it is impossible for all the minimal generators to have degrees 1 and 2 (as must be the case for (I, L)) and still be artinian. If $h_{R/(I,L)}(3) \leq 2$, it is impossible, by Macaulay's theorem, for the value to then rise to 3 in degree $e - 1$. Thus this value is constantly 3 for all $2 \leq i \leq e - 1$.

If $e - 1 \geq 4$, the growth from degree $e - 2$ to $e - 1$ is maximal, and again we get a contradiction from the artinian property and the generation in degrees ≤ 2 . The only remaining possibility is $e = 4$, which we now assume.

Recalling (5.2) and the precise list of monomials of degree $e - 1 = 3$ not in $\text{gin}(I)$ and not divisible by x_r , we observe that $\text{gin}\left(\frac{(I,L)}{(L)}\right)$ contains all the monomials of degree $e - 1 = 3$ in the variables x_1, \dots, x_{r-1} except the monomials x_{r-1}^3 , $x_{r-2}x_{r-1}^2$, and either $x_{r-2}^2x_{r-1}$ or $x_{r-3}x_{r-1}^2$. Consequently, in degree 2, $\text{gin}\left(\frac{(I,L)}{(L)}\right)$ contains all the monomials except one of the following sets of three:

- (1) x_{r-1}^2 , $x_{r-2}x_{r-1}$ and x_{r-2}^2 , or
- (2) x_{r-1}^2 , $x_{r-2}x_{r-1}$ and $x_{r-3}x_{r-1}$.

In case (2), one can check that then $\text{gin}\left(\frac{(I,L)}{(L)}\right)$ does not have any minimal generators of degree 3 (since the Hilbert function of the part in degree 2 grows to exactly 3 in degree 3). By the Crystallization Principle ([10] Proposition 2.28), since also $\frac{(I,L)}{(L)}$ has no minimal generators of degree 3, we see that $\left(\frac{(I,L)}{(L)}\right)_{\leq 2}$ defines a zero-dimensional scheme, rather than being artinian, which contradicts the assumption that R/I is presented by quadrics.

It remains to handle case (1). We notice first that $\text{gin}\left(\frac{(I,L)}{(L)}\right) \subset k[x_1, \dots, x_{r-1}]$, and that in degree 3 all three monomials not in $\text{gin}\left(\frac{(I,L)}{(L)}\right)$ are divisible by x_{r-1} . It follows that multiplication for the ring $R/(I, L)$ by a (new) general linear form is surjective from degree 2 to degree 3. However, we see that in case (1) the analogous multiplication from degree 1 to degree 2 has rank 2 rather than 3. Let us denote by $\bar{I} = \frac{(I,L)}{(L)} \subset \bar{R} = R/(L)$. Our observation implies that for a general linear form \bar{L} , the Hilbert function of $\bar{I} : \bar{L}$ is $(1, 2, 3)$. But the maximal growth from degree 1 to degree 2 precludes the degree 2 component of \bar{I} from being artinian. Since $\bar{I} \subset \bar{I} : \bar{L}$, the same is true of \bar{I} . And finally, then, the same is true of I . This contradiction shows that case (1) does not occur. \square

Corollary 5.14. *Let R/I be an artinian Gorenstein algebra presented by quadrics over $R = k[x_1, \dots, x_r]$ and having socle degree e . If $h_1 = h_2 = h_{e-2} = h_{e-1} = r$ then $e \leq \max\{3, \frac{r+6}{4}\}$.*

Proof. We showed in Proposition 5.2 that if the multiplication from degree 1 to degree 2 is an isomorphism then it must follow that $e = 3$. If the multiplication is not an injection and $e \geq 4$, then Theorem 5.13 implies $e \leq \frac{r+6}{4}$ as claimed. \square

We thus have, for small r , the result of Corollary 5.3 without assuming the Injectivity Conjecture:

Corollary 5.15. *Assume that $3 \leq r \leq 9$. Then the following are equivalent for an artinian Gorenstein algebra R/I presented by quadrics, with Hilbert function $\underline{h} = \{h_i, i \geq 0\}$, and such an algebra exists:*

- (1) $h_2 = r$
- (2) $e = 3$
- (3) $\underline{h} = (1, r, r, 1)$.

6. COMPUTER EVIDENCE

In Proposition 3.3 we gave a complete classification of the possible Hilbert functions of artinian Gorenstein algebras presented by quadrics, with $r \leq 5$. In this section we consider the next case, $r = 6$. By the results of Section 2 we know that the following Hilbert functions exist:

$$\begin{aligned} &(1, 6, 1) \\ &(1, 6, 6, 1) \\ &(1, 6, 10, 6, 1) \\ &(1, 6, 11, 6, 1) \\ &(1, 6, 13, 13, 6, 1) \\ &(1, 6, 15, 20, 15, 6, 1) \end{aligned}$$

The only question is whether other possibilities exist for $e = 4$, and the only value that is open is that for h_2 . We know that $h_2 \neq 14$ by Proposition 2.3. We also know that $h_2 \neq 6$ by Corollary 5.15. We first show that h_2 cannot be 13. Suppose it were. We have a minimal free resolution

$$\cdots \rightarrow R(-2)^8 \rightarrow I \rightarrow 0$$

and we know that we can link with a complete intersection of quadrics. The residual is thus level of type 2. But a Hilbert function computation gives that the residual has to have Hilbert function $(1, 6, 14, 14, 2)$, which is not a level sequence.

This still leaves several cases that we are not currently able to resolve. In this section we explore the existence and “number” of such algebras using CoCoA [5]. Interestingly, these seem to depend on the characteristic of the field, so perhaps new methods (that depend on the characteristic) will have to be developed to continue this study.

Let \mathfrak{c} be a complete intersection of quadrics in $k[x_1, \dots, x_6]$. Let F be a form of degree 2 not in \mathfrak{c} . Then $I = \mathfrak{c} : F$ defines an artinian Gorenstein algebra with socle degree 4, and I contains \mathfrak{c} . It may or may not be true that R/I is presented by quadrics. Conversely, if R/I is an artinian Gorenstein algebra with socle degree 4 such that I contains a complete intersection \mathfrak{c} of quadrics, then $\mathfrak{c} : I$ is an almost complete intersection generated by quadrics, $\mathfrak{c} : I = \mathfrak{c} + \langle F \rangle$. Then by linkage, $I = \mathfrak{c} : F$. Thus to study such algebras, we must study ideals of the form $\mathfrak{c} : F$ where \mathfrak{c} is a complete intersection of quadrics and $\deg F = 2$.

Clearly it is not possible to check all possible complete intersections together with all possible choices for F . For each of our searches, we first fixed the field, then we fixed \mathfrak{c} , and finally we let F vary over $[R]_2$. The fields that we used were \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_5 and \mathbb{Z}_7 . The

complete intersections that we used were either the monomial complete intersection or a randomly chosen one.

For the choices of F , our idea was to take as representative a sampling as possible. Over \mathbb{Z}_2 it was possible to do an exhaustive search. Indeed, note that R_2 has dimension 21, and a basis can be obtained whose first six elements are the six generators of the complete intersection. (In the monomial case this just means x_1^2, \dots, x_6^2 , while in the “random” case it is a set of six forms G_1, \dots, G_6 .) Without loss of generality, we can assume (but we checked on the computer in the case of the “random” choices) that the 15 squarefree monomials of degree 2 complete the six forms to obtain a basis of R_2 . This means that over \mathbb{Z}_2 , we can obtain all possible forms F modulo the complete intersection by looking at all possible sums of squarefree monomials. There are 32,767 such sums (the case $F = 0$ being trivial).

Over the other fields there are too many possibilities for F , so we did not do an exhaustive search. Instead, we randomly chose 30,000 forms of degree 2, checking each in turn. We followed this approach over \mathbb{Z}_2 as well, so that we could have some indication of the reliability of the results.

For each choice of F , we checked to see if the corresponding Gorenstein ideal was presented by quadrics. If so, we computed h_2 and kept track of it. The following tables show the results of these computations. The entries of the tables give the number of the resulting Gorenstein algebras that were presented by quadrics and had the corresponding value of h_2 .

We begin with the results over \mathbb{Z}_2 . It is not surprising that the monomial complete intersection gives different behavior from the one with randomly chosen generators, but it is an interesting comparison. The more meaningful comparisons are between the first and second column, and the third and fourth columns.

Results for $k = \mathbb{Z}_2$	Value of h_2	monomial comp. int., all F (/32,767)	monomial comp. int., random F (/30,000)	“random” comp. int., all F (/32,767)	“random” comp. int., random F (/30,000)
	10	18,228	16,753	2	4
	11	0	0	32	32
	12	0	0	572	531
	total	18,228	16,753	606	567

We remark that unfortunately we did not use the same “random” complete intersection for the third column as we did for the fourth column. Notice, in any case, that $\frac{18,228}{32,767} \approx 0.5563$ while $\frac{16,753}{30,000} \approx 0.5584$. Similarly, $\frac{572}{32,767} \approx 0.0175$ while $\frac{531}{30,000} \approx 0.0177$. So there is some hope that the proportions below are an accurate reflection of the true proportion.

In the next two tables, we give the results over \mathbb{Z}_3 , \mathbb{Z}_5 and \mathbb{Z}_7 . The first table gives the probabilistic results when \mathfrak{c} is the monomial complete intersection, while the second gives the probabilistic results when \mathfrak{c} is a randomly chosen complete intersection. In both cases, we checked 30,000 randomly chosen quadratic forms F to obtain our artinian Gorenstein algebras as quotients of R/\mathfrak{c} .

Monomial Complete Intersection	Value of h_2	$k = \mathbb{Z}_3$	$k = \mathbb{Z}_5$	$k = \mathbb{Z}_7$
	10	800	49	15
	11	31	1	0
	12	89	13	10
	total	920	63	25

Random Complete Intersection	Value of h_2	$k = \mathbb{Z}_3$	$k = \mathbb{Z}_5$	$k = \mathbb{Z}_7$
	10	0	0	0
	11	0	0	0
	12	44	2	0
	total	44	2	0

One notes that in each case, of the 30,000 algebras constructed, very few are presented by quadrics (except over \mathbb{Z}_2), and they get sparser as the characteristic grows. But perhaps the most interesting aspect of this list is the observation that we have not been able to directly construct examples with $h_2 = 12$ theoretically, while the computer shows that they exist but become more and more rare as the characteristic grows. One wonders if they in fact exist at all in characteristic zero. It is also surprising that the values of h_2 that we know do exist did not arise in the latter table (for the randomly chosen complete intersection). Similarly, one wonders if in characteristic zero (or even sufficiently large characteristic) there is an open subset of $[R]_2$ corresponding to values of F with the property that none of the corresponding algebras have quotients presented by quadrics.

Acknowledgements: We thank Gunnar Fløystad for an interesting discussion which led us to study the problem considered here, and Fabrizio Zanello for some interesting related discussions.

REFERENCES

- [1] J. Ahn and J. Migliore, *Some geometric results arising from the Borel fixed property*, J. Pure and Applied Algebra 209 (2007), 337–360.
- [2] J. Ahn and Y. Shin, *Generic initial ideals and graded Artinian-level algebras not having the Weak-Lefschetz Property*, J. Pure and Applied Algebra 210 (2007), 855–879.
- [3] T. Ananyan and M. Hochster, *Ideals Generated by Quadratic Polynomials*, Preprint, 2011.
- [4] D. Bernstein and A. Iarrobino: *A non-unimodal graded Gorenstein Artin algebra in codimension five*, Comm. in Algebra **20** (1992), No. 8, 2323–2336.
- [5] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra, Available at <http://cocoa.dima.unige.it>
- [6] E. Davis, A.V. Geramita, F. Orecchia, *Gorenstein Algebras and the Cayley-Bacharach Theorem*, Proc. Amer. Math. Soc. **93** (1985), 593–597.
- [7] D. Eisenbud, M. Green and J. Harris, *Cayley-Bacharach theorems and conjectures*, Bull. Amer. Math. Soc. **33** (1996), 295–324.
- [8] L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), 51–67.
- [9] A.V. Geramita, *Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995), 2–114, Queen’s Papers in Pure and Appl. Math., 102, Queen’s Univ., Kingston, ON, 1996.
- [10] M. Green, *Generic Initial Ideals*, in: “Six Lectures on Commutative Algebra,” 119–186, Progress in Math. 166, Birkhäuser, Basel, 1998.
- [11] T. Harima, J. Migliore, U. Nagel, and J. Watanabe: *The Weak and Strong Lefschetz Properties for artinian K -Algebras*, J. Algebra **262** (2003), 99–126.

- [12] C. Huneke, B. Ulrich, *General Hyperplane Sections of Algebraic Varieties*, J. Alg. Geom. **2** (1993), 487–505.
- [13] A. Iarrobino and V. Kanev, “Power sums, Gorenstein algebras, and determinantal loci.” Appendix C by Iarrobino and Steven L. Kleiman. Lecture Notes in Mathematics **1721**, Springer-Verlag, Berlin, 1999.
- [14] E. Kunz, *Almost complete intersections are not Gorenstein rings*, J. Algebra **28** (1974), 111–115.
- [15] J. Migliore, “Introduction to Liaison Theory and Deficiency Modules,” Birkhäuser, Progress in Mathematics **165**, Boston, 1998.
- [16] J. Migliore and U. Nagel, *Liaison and related topics: Notes from the Torino Workshop-School*, Rend. Sem. Mat. Univ. Pol. Torino **59**, 2 (2001), 59–126.
- [17] J. Migliore and F. Zanello, *The strength of the Weak Lefschetz Property*, Illinois J. Math. **52** (2008), no. 4, 1417–1433.
- [18] J. Migliore, R. Miró-Roig and U. Nagel, *Monomial almost complete intersections and the Weak Lefschetz Property*, Trans. Amer. Math. Soc. **363**, No. 1 (2011), 229–257.
- [19] J. Migliore, R. Miró-Roig and U. Nagel, *On the Weak Lefschetz Property for Powers of Linear Forms*, Preprint, 2010.
- [20] U. Nagel and Y. Pitteloud, *On graded Betti numbers and geometrical properties of projective varieties*, manuscripta math. **84** (1994), 291–314.
- [21] L. Reid, L. Roberts and M. Roitman, *On complete intersections and their Hilbert functions*, Canad. Math. Bull. **34** (4) (1991), 525–535.
- [22] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184.
- [23] R. Stanley, *Combinatorics and Commutative Algebra*, 2nd edition. Progress in Mathematics **41** Birkhäuser Boston, Inc. Boston, MA, 1996.
- [24] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. 11, Kinokuniya Co. North Holland, Amsterdam (1987), 303–312.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556, USA
E-mail address: Juan.C.Migliore.1@nd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, 715 PATTERSON OFFICE TOWER,
 LEXINGTON, KY 40506-0027, USA
E-mail address: uwe.nagel@uky.edu